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# Interval Type-2 Fuzzy-Model-Based Control Design for Systems Subject to Actuator Saturation under Imperfect Premise Matching

Yuandi Li and H. K. Lam

**Abstract**— In this paper, the problems of asymptotic stabilization for interval type-2 fuzzy systems with actuator saturation are investigated. The sufficient conditions for the existence of the interval type-2 fuzzy controller are in terms of sum-of-squares (SOS). The problem is formulated and solved with more flexibility due to imperfect premise matching that the number of rules and premise membership functions of the fuzzy controller are not necessarily the same as the ones of the fuzzy model. The actuator saturation is depicted and dealt with contractively invariant ellipsoid. Piecewise linear membership functions enclosing the original lower and upper membership functions are employed to facilitate the stability analysis. A numerical example indicates the effectiveness of the derived results.

## I. INTRODUCTION

TYPE-2 fuzzy systems have been widely studied in both theory and practice during the last decade. [1]–[8] are just a very small part of the fruitful results on the study of type-2 fuzzy systems. In 1975, Zadeh [9] introduced the type-2 fuzzy systems for the first time. One of the many motivations for studying such a class of systems is that type-2 fuzzy sets are better in representing and capturing uncertainties [10], [11]. Because type-1 fuzzy sets do not contain uncertain information, they can not deal with the nonlinear plant which suffers the parameter uncertainties. Type-2 and type-1 fuzzy systems are both characterized by IF-THEN rules and represented as weighted sum of local linear systems. In a more general sense, type-2 fuzzy systems could be regarded as a bunch of type-1 fuzzy systems.

However, the original type-2 fuzzy set was not designed for fuzzy-model-based control framework. This stimulates the study on interval type-2 fuzzy model which describes the nonlinear plant subject to parameter uncertainty captured by upper and lower membership functions. Interval type-2 fuzzy systems were first proposed in [12] and then extended in [13] for a wider class of nonlinear systems. Since then, they have found lots of successful applications such as energy markets, face recognition, image processing, supervisory adaptive tracking control, linguistic summarization and so on [14]–[18]. Preliminary stability results on interval type-2 fuzzy-model-based systems could be found in [13] under parallel distributed compensation (PDC) scheme. [19] further reduces implementation complexity and increases design flexibility by considering the imperfect premise matching.

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On the other hand, it is well known that most practical dynamic systems involve actuator saturation as physical capacity of the actuator is always limited. Without taking the limitations into consideration, techniques developed may result in severe performance degradation or even instability of the closed-loop control system. In recent years, fuzzy system with actuator saturation has been probed widely. Just to name a few, in [20], the authors proposed robust stability analysis and fuzzy-scheduling control for nonlinear system by maximizing the domain of attraction of a T-S fuzzy system. In [21], the authors investigated  $H_\infty$  control problem subject to actuator saturation for nonlinear systems by the fuzzy scheme. In [22], the authors dealt with performance constrained control problem for nonlinear stochastic systems subject to  $H_\infty$  performance constraint and actuator saturation.

In this paper we investigate interval type-2 fuzzy-model-based control design for systems with actuator saturation under imperfect premise matching. Unlike existing work under type-1 fuzzy logic frame, this paper formulates nonlinear systems with parameter uncertainties and actuator saturation under interval type-2 fuzzy logic frame. The uncertainties are presented by the upper and lower membership functions and the actuator saturation is dealt with invariant set theory.

The rest of this paper is organized as follows: Section II is devoted to the mathematical description of the concerned system and some preliminaries. Synthesis of saturated state-feedback type-2 fuzzy controller are presented in Section III. A numerical simulation in Section IV is to illustrate the feasibility and effectiveness of the proposed results. Finally, Section V concludes the paper.

## II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a nonlinear system with actuator saturation and parameter uncertainties represented by the following interval type-2 fuzzy model with lower and upper bound membership functions.

**Plant Rule  $i$ :**

IF  $\theta_1(x(t))$  is  $\tilde{M}_{i1}$ ,  $\theta_2(x(t))$  is  $\tilde{M}_{i2}$   $\cdots$  and  $\theta_\Psi(x(t))$  is  $\tilde{M}_{i\Psi}$ , THEN

$$\dot{x}(t) = A_i x(t) + B_i \bar{u}(t) \quad (1)$$

where  $\tilde{M}_{i\alpha}$  is an interval type-2 fuzzy set of rule  $i$ ,  $\alpha = 1, 2, \dots, \Psi$  and  $i = 1, 2, \dots, p$ .  $x(t) \in \mathbb{R}^n$  is the state,  $\bar{u}(t) = [\bar{u}_1(t), \dots, \bar{u}_m(t)]^T = [\text{sat}(u_1(t)), \dots, \text{sat}(u_m(t))]^T = \text{sat}(u(t)) \in \mathbb{R}^m$  is the saturated control input.  $A_i$ ,  $B_i$ , are known matrices as system matrices and input matrices

respectively. The firing strength of rule  $i$  is the interval sets as follows:

$$W_i(x(t)) = [\underline{w}_i(x(t)), \overline{w}_i(x(t))], \quad i = 1, 2, \dots, p \quad (2)$$

where

$$\underline{w}_i(x(t)) = \prod_{\alpha=1}^{\Psi} \underline{\mu}_{\tilde{M}_{i\alpha}}(\theta_{\alpha}(x(t))) \geq 0 \quad (3)$$

$$\overline{w}_i(x(t)) = \prod_{\alpha=1}^{\Psi} \overline{\mu}_{\tilde{M}_{i\alpha}}(\theta_{\alpha}(x(t))) \geq 0 \quad (4)$$

in which  $\underline{\mu}_{\tilde{M}_{i\alpha}}(\theta_{\alpha}(x(t)))$  and  $\overline{\mu}_{\tilde{M}_{i\alpha}}(\theta_{\alpha}(x(t)))$  denote the lower and upper membership functions respectively satisfying the property  $\overline{\mu}_{\tilde{M}_{i\alpha}}(\theta_{\alpha}(x(t))) \geq \underline{\mu}_{\tilde{M}_{i\alpha}}(\theta_{\alpha}(x(t))) \geq 0$  and  $\underline{w}_i(x(t))$  and  $\overline{w}_i(x(t))$  denote the lower and upper grade of membership respectively. The inferred interval type-2 fuzzy model is defined as follows:

$$\dot{x}(t) = \sum_{i=1}^p \tilde{w}_i(x(t))(A_i x(t) + B_i \bar{u}(t)) \quad (5)$$

where

$$\tilde{w}_i(x(t)) = \underline{w}_i(x(t))\underline{\alpha}_i(x(t)) + \overline{w}_i(x(t))\overline{\alpha}_i(x(t)) \geq 0 \quad \forall i \quad (6)$$

$$\sum_{i=1}^p \tilde{w}_i(x(t)) = 1 \quad (7)$$

in which  $\underline{\alpha}_i(x(t)) \in [0, 1]$ ,  $\overline{\alpha}_i(x(t)) \in [0, 1]$  are nonlinear functions with the property that  $\underline{\alpha}_i(x(t)) + \overline{\alpha}_i(x(t)) = 1$ . **Definition 1:** The actuator saturation could be defined as follows:

$$\bar{u}_l(t) = \text{sat}(u_l) = \begin{cases} u_{lH} & u_{lH} < u_l \\ u_l & u_{lL} \leq u_l \leq u_{lH} \\ u_{lL} & u_l < u_{lL} \end{cases} \quad (8)$$

where  $l = 1, 2, \dots, m$  and  $u_{lH} > 0 > u_{lL}$ .

We introduce a parameter  $\epsilon$ , which is  $0 < \epsilon < 1$ , to make sure the saturation map  $\text{sat}(\cdot)$  is inside the sector  $(\epsilon, 1)$ . Then we have

$$\frac{u_{lL}}{\epsilon} \leq u_l \leq \frac{u_{lH}}{\epsilon} \quad (9)$$

For simplicity, if we set  $u_{lH} = -u_{lL}$ , then we have

$$|u_l| \leq \frac{u_{lH}}{\epsilon} \quad (10)$$

With all the settings and definition above, one can get

$$\|\bar{u}(t) - \frac{1+\epsilon}{2}u(t)\| \leq \frac{1-\epsilon}{2}\|u(t)\| \quad (11)$$

#### Controller Rule $j$ :

IF  $\sigma_1(x(t))$  is  $\tilde{N}_{j1}$ ,  $\sigma_2(x(t))$  is  $\tilde{N}_{j2} \dots$  and  $\sigma_{\Omega}(x(t))$  is  $\tilde{N}_{j\Omega}$ , THEN

$$u(t) = K_j x(t) \quad (12)$$

where  $\tilde{N}_{j\beta}$  is an interval type-2 fuzzy set of rule  $j$ ,  $\beta = 1, 2, \dots, \Omega$  and  $j = 1, 2, \dots, c$ .  $K_j$  are unknown feedback

gains to be determined. The firing strength of rule  $j$  is the interval sets as follows:

$$M_j(x(t)) = [\underline{m}_j(x(t)), \overline{m}_j(x(t))], \quad j = 1, 2, \dots, c \quad (13)$$

where

$$\underline{m}_j(x(t)) = \prod_{\beta=1}^{\Omega} \underline{\mu}_{\tilde{N}_{j\beta}}(\sigma_{\beta}(x(t))) \geq 0 \quad (14)$$

$$\overline{m}_j(x(t)) = \prod_{\beta=1}^{\Omega} \overline{\mu}_{\tilde{N}_{j\beta}}(\sigma_{\beta}(x(t))) \geq 0 \quad (15)$$

in which  $\underline{\mu}_{\tilde{N}_{j\beta}}(\sigma_{\beta}(x(t)))$  and  $\overline{\mu}_{\tilde{N}_{j\beta}}(\sigma_{\beta}(x(t)))$  denote the lower and upper membership functions respectively satisfying the property  $\overline{\mu}_{\tilde{N}_{j\beta}}(\sigma_{\beta}(x(t))) \geq \underline{\mu}_{\tilde{N}_{j\beta}}(\sigma_{\beta}(x(t))) \geq 0$  and  $\underline{m}_j(x(t))$  and  $\overline{m}_j(x(t))$  denote the lower and upper grade of membership respectively. The inferred interval type-2 fuzzy controller is defined as follows:

$$u(t) = \sum_{j=1}^c \tilde{m}_j(x(t))K_j x(t) \quad (16)$$

where

$$\tilde{m}_j(x(t)) = \frac{\underline{m}_j(x(t))\underline{\beta}_j(x(t)) + \overline{m}_j(x(t))\overline{\beta}_j(x(t))}{\sum_{k=1}^c (\underline{m}_k(x(t))\underline{\beta}_k(x(t)) + \overline{m}_k(x(t))\overline{\beta}_k(x(t)))} \geq 0 \quad \forall j \quad (17)$$

$$\sum_{j=1}^c \tilde{m}_j(x(t)) = 1 \quad (18)$$

in which  $\underline{\beta}_j(x(t)) \in [0, 1]$ ,  $\overline{\beta}_j(x(t)) \in [0, 1]$  are predefined functions with the property that  $\underline{\beta}_j(x(t)) + \overline{\beta}_j(x(t)) = 1$ .

With the plant and controller expressions and the properties of  $\sum_{i=1}^p \tilde{w}_i(x(t)) = 1$ ,  $\sum_{j=1}^c \tilde{m}_j(x(t)) = 1$ ,  $\sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(x(t))\tilde{m}_j(x(t)) = 1$ , we can have the closed-loop control system as

$$\dot{x}(t) = \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(x(t))((A_i + \frac{1+\epsilon}{2}B_i K_j)x(t) + B_i R) \quad (19)$$

where  $\tilde{h}_{ij}(x(t)) \triangleq \tilde{w}_i(x(t))\tilde{m}_j(x(t))$ ,  $R = \bar{u}(t) - \frac{1+\epsilon}{2}u(t)$ .

In addition  $\tilde{h}_{ij}(x(t))$  could be reconstructed as  $\bar{\gamma}_{ij}(x(t))\bar{h}_{ij}(x(t)) + \underline{\gamma}_{ij}(x(t))\underline{h}_{ij}(x(t))$ , in which  $\underline{\gamma}_{ij}(x(t)) \in [0, 1]$ ,  $\bar{\gamma}_{ij}(x(t)) \in [0, 1]$  are functions with the property that  $\underline{\gamma}_{ij}(x(t)) + \bar{\gamma}_{ij}(x(t)) = 1$  and  $\bar{h}_{ij}(x(t))$  and  $\underline{h}_{ij}(x(t))$  are the piecewise linear membership function approximations of the upper and lower bound of  $\tilde{h}_{ij}(x(t))$  with definitions below from [19]

$$\bar{h}_{ij}(x(t)) = \sum_{k=1}^q \sum_{i_1=1}^2 \dots \sum_{i_n=1}^2 \prod_{r=1}^n v_{r i_r k}(x_r(t)) \bar{\delta}_{ij i_1 i_2 \dots i_n k} \quad (20)$$

$$\underline{h}_{ij}(x(t)) = \sum_{k=1}^q \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r,k}(x_r(t)) \delta_{ij i_1 i_2 \dots i_n k} \quad (21)$$

where  $0 \leq \underline{\delta}_{ij i_1 i_2 \dots i_n k} \leq \bar{\delta}_{ij i_1 i_2 \dots i_n k} \leq 1$  are scalars to be figured out,  $0 \leq \underline{h}_{ij}(x(t)) \leq \bar{h}_{ij}(x(t)) \leq 1$ ,  $v_{ri_r,k}(x_r(t)) \in [0, 1]$  and  $v_{r1k}(x_r(t)) + v_{r2k}(x_r(t)) = 1$ , otherwise  $v_{ri_r,k}(x_r(t)) = 0$ ,  $x(t) \in \Psi_k$ ,  $\cup_{k=1}^q \Psi_k = \Psi$  is the state space of interest.

*Remark 1:* With the above definitions, in the further stability analysis, we could use scalars  $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$  and  $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$  to deal with the term  $\underline{h}_{ij}(x(t))$  and  $\bar{h}_{ij}(x(t))$  through  $\prod_{r=1}^n v_{ri_r,k}(x_r(t))$  which are independent of  $i$  and  $j$ . In a word, the stability conditions involving the membership function information ( $\bar{h}_{ij}(x(t))$  and  $\underline{h}_{ij}(x(t))$  as the upper and lower bound of  $\tilde{h}_{ij}(x(t))$ ) could be achieved by scalars  $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$  and  $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$ .

*Definition 2:* Let an ellipsoid  $\Lambda_1$  and a positive scalar function  $V(x(t))$  be defined as follows, respectively,

$$\Lambda_1 = \{x(t) | x^T(t) P x(t) \leq 1\} \quad (22)$$

and

$$V(x(t)) = x^T(t) P x(t) \quad (23)$$

where  $P \in \mathbb{R}^{n \times n}$  denotes a positive definite matrix. An ellipsoid  $\Lambda_1$ , which is inside the domain of attraction, is said to be contractively invariant [20] if

$$\dot{V}(x(t)) < 0, \forall x(t) \in \Lambda_1 \setminus \{0\}$$

The saturation constraint on the input  $u_l$  could be expressed as follows

$$|\sum_{j=1}^c \tilde{m}_j(x(t)) K_j^{(l)} x(t)| \leq \frac{u_{lH}}{\epsilon} \quad (24)$$

where  $K_j^{(l)}$  denotes the  $l$ th row of  $K_j$ . Let

$$\Lambda_2 = \left\{x(t) | x^T(t) (K_j^{(l)})^T (K_j^{(l)}) x(t) \leq \left(\frac{u_{lH}}{\epsilon}\right)^2 \right\} \quad (25)$$

It is required that  $x(t) \in \Lambda_1 \subset \Lambda_2$ . The equivalent condition is

$$(K_j^{(l)}) P^{-1} (K_j^{(l)})^T \leq \left(\frac{u_{lH}}{\epsilon}\right)^2 \quad (26)$$

### III. MAIN RESULTS

For simplification reason, we denote  $\tilde{w}_i(x(t))$ ,  $\tilde{m}_j(x(t))$ ,  $\tilde{h}_{ij}(x(t))$ ,  $\bar{h}_{ij}(x(t))$  and  $\underline{h}_{ij}(x(t))$  as  $\tilde{w}_i$ ,  $\tilde{m}_j$ ,  $\tilde{h}_{ij}$ ,  $\bar{h}_{ij}$  and  $\underline{h}_{ij}$ , respectively.

We need to revisit two lemmas to be used in the following proof for the sufficient stability conditions.

*Lemma 1* [23]: For any two matrices  $X$  and  $Y$ , one has

$$X^T Y + Y^T X \leq X^T X + Y^T Y$$

*Lemma 2* [24]: Suppose that matrices  $M_i \in \mathbb{R}^{m \times n}$ ,  $i = 1, 2, \dots, r$ , and a positive matrix  $Q \in \mathbb{R}^{m \times m}$  are given. If  $\sum_{i=1}^r p_i = 1$  and  $0 \leq p_i \leq 1$ , then

$$\left(\sum_{i=1}^r p_i M_i\right)^T Q \left(\sum_{i=1}^r p_i M_i\right) \leq \sum_{i=1}^r p_i M_i^T Q M_i$$

*Theorem 1:* Given predefined scalars  $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$  and  $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$  satisfying (20) and (21), if there exist matrices  $X = X^T$ ,  $Y_{ij} = Y_{ij}^T$ ,  $\Omega_{ij}$  and  $N_j$  of appropriate dimensions such that the following SOS-based conditions hold:

$$\begin{aligned} v_1^T (X - \varepsilon_1 I) v_1 & \text{ is SOS} \\ v_2^T (Y_{ij} - \varepsilon_2 I) v_2 & \text{ are SOS} \\ v_2^T (Y_{ij} - \Omega_{ij} - \varepsilon_3 I) v_2 & \text{ are SOS} \\ -v_2^T \left( \sum_{i=1}^p \sum_{j=1}^c (\delta_{ij i_1 i_2 \dots i_n k} \Omega_{ij} \right. \\ & \left. + (\bar{\delta}_{ij i_1 i_2 \dots i_n k} - \delta_{ij i_1 i_2 \dots i_n k}) Y_{ij} + \varepsilon_4 I) v_2 \right) & \text{ are SOS} \\ v_2^T \left( \begin{bmatrix} \left(\frac{1+\epsilon}{2} \cdot \frac{u_{lH}}{\epsilon}\right)^2 & \frac{1+\epsilon}{2} N_j^{(l)} \\ * & X \end{bmatrix} - \varepsilon_5 I \right) v_2 & \text{ are SOS} \end{aligned} \quad (27)$$

where

$$\Omega_{ij} = \begin{bmatrix} W_{ij} + B_i B_i^T & \frac{1-\epsilon}{2} N_j^T \\ * & -I \end{bmatrix} \quad (28)$$

$W_{ij} = A_i X + \frac{1+\epsilon}{2} B_i N_j + (A_i X + \frac{1+\epsilon}{2} B_i N_j)^T$ ,  $i = 1, 2, \dots, p$ ,  $j = 1, 2, \dots, c$ ,  $l = 1, 2, \dots, m$ .  $N_j^{(l)}$  denotes the  $l$ th row of  $N_j$ .  $v_1, v_2$  are arbitrary vectors independent of  $x$ .  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$  and  $\varepsilon_5$  are predefined positive scalars. Then the closed-loop control system (19) is asymptotically stable. In addition, the interval type-2 fuzzy controller gains can be obtained by  $K_j = N_j X^{-1}$ .

*Proof:* Consider a candidate of Lyapunov-Krasovskii functional as the one described in Definition 2

$$V(x(t)) = x^T(t) P x(t)$$

Along the trajectories of the closed-loop control system, the corresponding time derivative of  $V(x(t))$  is given by

$$\begin{aligned} \dot{V}(x(t)) &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) \left[ (A_i + \frac{1+\epsilon}{2} B_i K_j)^T P \right. \\ &\quad \left. + P (A_i + \frac{1+\epsilon}{2} B_i K_j) \right] x(t) \\ &\quad + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j [R^T B_i^T P x(t) + x^T(t) P B_i R] \\ &\leq \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) \left[ (A_i + \frac{1+\epsilon}{2} B_i K_j)^T P \right. \\ &\quad \left. + P (A_i + \frac{1+\epsilon}{2} B_i K_j) \right] x(t) \\ &\quad + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j [x^T(t) P B_i B_i^T P x(t) + R^T R] \\ &\leq \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) \left[ (A_i + \frac{1+\epsilon}{2} B_i K_j)^T P \right. \\ &\quad \left. + P (A_i + \frac{1+\epsilon}{2} B_i K_j) \right] x(t) \\ &\quad + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) P B_i B_i^T P x(t) \\ &\quad + \left(\frac{1-\epsilon}{2}\right)^2 \left(\sum_{j=1}^c \tilde{m}_j K_j x(t)\right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) [(A_i + \frac{1+\epsilon}{2} B_i K_j)^T P \\
&\quad + P(A_i + \frac{1+\epsilon}{2} B_i K_j)] x(t) \\
&\quad + \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) P B_i B_i^T P x(t) \\
&\quad + (\frac{1-\epsilon}{2})^2 \sum_{j=1}^c \tilde{m}_j x^T(t) K_j^T K_j x(t) \\
&= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i \tilde{m}_j x^T(t) [(A_i + \frac{1+\epsilon}{2} B_i K_j)^T P \\
&\quad + P(A_i + \frac{1+\epsilon}{2} B_i K_j) \\
&\quad + P B_i B_i^T P + (\frac{1-\epsilon}{2})^2 K_j^T K_j] x(t)
\end{aligned}$$

By using Schur Complement and setting  $X^{-1} = P$ , we can rewrite the stability condition as

$$M(x(t)) = \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij} x^T(t) \Omega_{ij} x(t) < 0 \quad (29)$$

In order to include the information of the membership functions, we introduce slack matrices  $Y_{ij}$  into the analysis with the property  $Y_{ij} > 0$ ,  $Y_{ij} > \Omega_{ij}$ . Then we have

$$\begin{aligned}
M(x(t)) &= \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij} x^T(t) \Omega_{ij} x(t) \\
&= \sum_{i=1}^p \sum_{j=1}^c (\underline{h}_{ij} + (\tilde{h}_{ij} - \underline{h}_{ij})) x^T(t) \Omega_{ij} x(t) \\
&\leq \sum_{i=1}^p \sum_{j=1}^c \underline{h}_{ij} x^T(t) \Omega_{ij} x(t) \\
&\quad + (\tilde{h}_{ij} - \underline{h}_{ij}) x^T(t) Y_{ij} x(t) \\
&= \sum_{i=1}^p \sum_{j=1}^c x^T(t) (\underline{h}_{ij} \Omega_{ij} + (\tilde{h}_{ij} - \underline{h}_{ij}) Y_{ij}) x(t)
\end{aligned}$$

Recalling  $\sum_{k=1}^q \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \prod_{r=1}^n v_{ri_r k}(x_r(t)) = 1$ , with (20) and (21) and the constraint on the input, we can convert the above conditions into the SOS-based ones stated in the theorem. The proof is complete.

*Remark 2:* Theorem 1 introduces membership functions  $\tilde{h}_{ij}$  which are reconstructed by the upper bound  $\bar{h}_{ij}$  and lower bound  $\underline{h}_{ij}$ . Moreover  $\bar{h}_{ij}$  and  $\underline{h}_{ij}$  could be expressed by predefined scalars  $\bar{\delta}_{ij i_1 i_2 \dots i_n k}$  and  $\underline{\delta}_{ij i_1 i_2 \dots i_n k}$  in the form of (20) and (21). This will allow us to just check conditions at certain points rather than every point of the membership functions.

*Remark 3:* As Theorem 1 involves the information of the membership functions in control design, it is a membership function dependent method which is less conservative than the membership function independent method. While Theorem 1 could be reduced to the basic stability conditions using

the membership function independent method for control design.

#### IV. NUMERICAL EXAMPLE

To show the effectiveness and efficiency of the proposed theoretical results, the following simulations are performed.

Consider a three-rule interval type-2 fuzzy model in the form of (1) with  $A_1 = \begin{bmatrix} 0 & 1 \\ 28.82 & -0.039 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 0 & 1 \\ 26.71 & -0.037 \end{bmatrix}$ ,  $A_3 = \begin{bmatrix} 0 & 1 \\ 22.07 & -0.033 \end{bmatrix}$ ,  $B_1 = [0 \ -3.27]^T$ ,  $B_2 = [0 \ -2.90]^T$ ,  $B_3 = [0 \ -2.09]^T$ .  $x = [x_1 \ x_2]^T$ .  $U_{IH} = -U_{IL} = 5$ ,  $\epsilon = 0.5$ .

The membership functions for the plant (1) are chosen as  $\tilde{w}_1(x_1) = 1 - 1/(1 + e^{-(x_1+4+\eta(t))})$ ,  $\tilde{w}_2(x_1) = 1 - \tilde{w}_1(x_1) - \tilde{w}_3(x_1)$ ,  $\tilde{w}_3(x_1) = 1/(1 + e^{-(x_1-4-\eta(t))})$ . Due to parameter uncertainty  $\eta(t) \in (-0.1, 0.1)$ , the membership functions are uncertain grades of membership. The lower and upper membership functions to be approximated by the piecewise linear membership functions for the three-rule plant (1) are chosen as  $\underline{w}_1(x_1) = 1 - 1/(1 + e^{-(x_1+4+0.25)})$ ,  $\bar{w}_2(x_1) = 1 - \underline{w}_1(x_1) - \underline{w}_3(x_1)$ ,  $\underline{w}_3(x_1) = 1/(1 + e^{-(x_1-4-0.25)})$ ,  $\bar{w}_1(x_1) = 1 - 1/(1 + e^{-(x_1+4-0.25)})$ ,  $\bar{w}_2(x_1) = 1 - \bar{w}_1(x_1) - \bar{w}_3(x_1)$ ,  $\bar{w}_3(x_1) = 1/(1 + e^{-(x_1-4+0.25)})$ . The lower and upper membership functions to be approximated by the piecewise linear membership functions for the two-rule controller (16) are chosen as  $\underline{m}_1(x_1) = 1 - 1/e^{-(x_1+0.25)/2}$ ,  $\bar{m}_1(x_1) = 1 - 1/e^{-(x_1-0.25)/2}$ ,  $\underline{m}_2(x_1) = 1 - \bar{m}_1(x_1)$ ,  $\bar{m}_2(x_1) = 1 - \underline{m}_1(x_1)$ . From (17), we can get  $\tilde{m}_j(x_1)$  by setting  $\bar{\beta}_1 = \bar{\beta}_2 = 0.5$ .

The stability conditions in Theorem 1 are employed to provide the feedback gains. We consider the grades of membership are capped and focus on the region  $x_1 \in [-10, 10]$ . The sample points of  $\underline{h}_{ij}$  and  $\bar{h}_{ij}$  are set as  $x_1 = \{-10, -9, \dots, 9, 10\}$ . Fig. 1 shows the membership function information of  $\tilde{h}_{ij}$  with solid lines and  $\bar{h}_{ij}$  and  $\underline{h}_{ij}$  with dash-dot lines. We choose  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0.001$ . The solution could be found by using third-party MATLAB toolbox SOSTOOLS [25].

With the above settings, the controller gains are obtained as  $K_1 = [47.14 \ 9.17]$ ,  $K_2 = [48.11 \ 9.09]$  and  $P = \begin{bmatrix} 29.93 & 6.36 \\ 6.36 & 1.89 \end{bmatrix}$ . Fig. 2 gives the state response of the closed-loop control system which is asymptotically stable with the initial state  $x(0) = [0.2, -0.1]^T$ . Fig. 3 shows the constrained control input with the initial state  $x(0) = [0.2, -0.1]^T$ . Fig. 4 shows the contractively invariant ellipsoid  $\Lambda_1$  and the trajectories of different initial states.

#### V. CONCLUSIONS

The stability of interval type-2 fuzzy-model-based control systems with actuator saturation is investigated in this paper. We have proposed a saturated interval type-2 fuzzy state-feedback controller to ensure the asymptotic stability of the closed-loop control system under imperfect premise matching. More design flexibility could be achieved, because it is not required that the fuzzy controller and fuzzy plant have

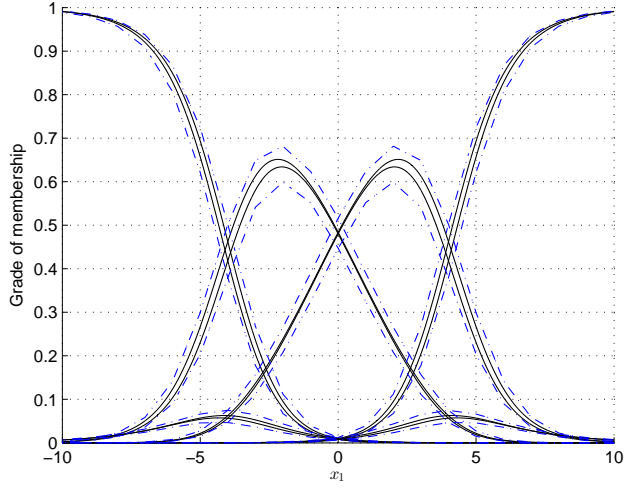


Fig. 1. Membership function information with three-rule model and two-rule controller

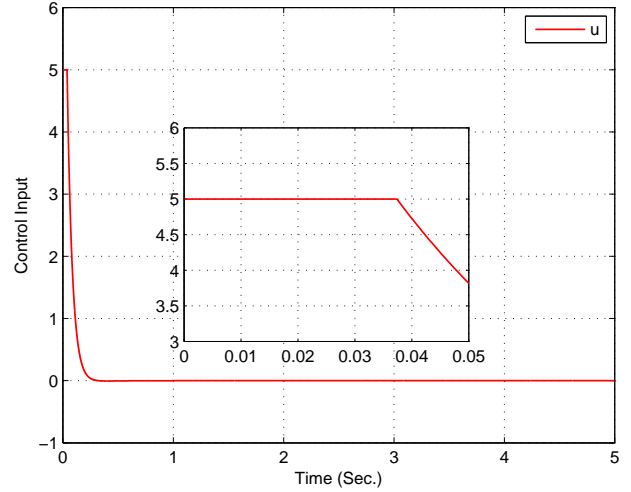


Fig. 3. Constrained control input of the closed-loop control system with  $x(0) = [0.2, -0.1]^T$

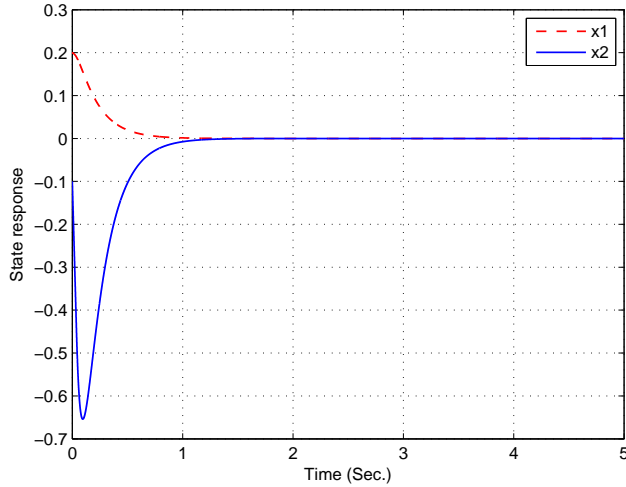


Fig. 2. State response of the closed-loop control system with  $x(0) = [0.2, -0.1]^T$

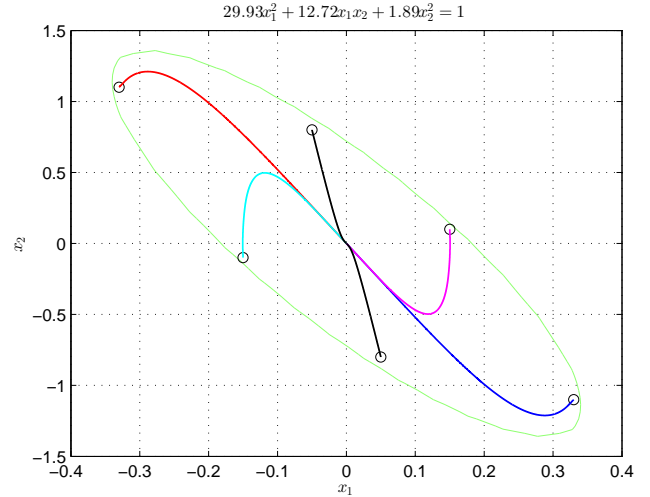


Fig. 4. State trajectories of the closed-loop control system with different initial states and the contractively invariant ellipsoid  $\Lambda_1$

the same premise membership function and/or number of fuzzy rules. The stability conditions based on the invariant ellipsoid come in SOS form and include the information of the membership functions to be more relaxed than membership function independent method. A numerical example is presented to show the effectiveness of the proposed approach.

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